Termination of planetary accretion due to gap formation.

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ABSTRACT

The process of gap formation by a growing planetary embryo embedded in a planetesimal disk is considered. It is shown that there exists a single parameter characterizing this process, which represents the competition between the gravitational influence of the embryo and planetesimal-planetesimal scattering. For realistic assumptions about the properties of the planetesimal disk and the planetary embryo, a gap is opened long before the embryo can accrete all the bodies within its region of influence. The implication of this result is that the embryo stops growing and, thus, large bodies formed during the coagulation stage should be less massive than is usually assumed. For conditions expected at 1 AU in the solar protoplanetary disk, gap formation is expected to occur around bodies of mass $\lesssim 10^{24}$ g. The effect of protoplanetary radial migration is also discussed.

Subject headings: planets and satellites: general — solar system: formation — (stars:) planetary systems

1. Introduction.

The formation of planets is one of the most complex problems in astrophysics, involving accumulation of bodies over some 45 orders of magnitude in mass — from dust grains to giant planets.

One issue which has received a lot of attention is the formation of planetary embryos by the accretion of planetesimals. From the perspective of dynamics we call an object an embryo when it becomes so massive that one can no longer describe its behavior by means of simple kinetic theory (in this paper we use names embryo, protoplanet and massive body interchangeably). In other words, in the presence of embryos the multiparticle distribution function cannot be taken as a product of one-particle distribution functions; the gravitational influence of the embryo is strong enough to affect the distribution of planetesimals with which it interacts. For example, a gap could form around the embryo. To study properties of systems containing embryos one must either resort to N-body simulations or try to account properly for their influence on the underlying planetesimal population and on each other.

In the standard scenario, protoplanets grow in orderly (Safronov, 1972) or runaway fashion (Wetherill & Stewart, 1989; Wetherill & Stewart, 1993) by accreting planetesimals from the protoplanetary nebula. After the largest bodies become embryos, they open gaps and accretion slows or stops. Thereafter these massive bodies evolve more slowly as gravitational encounters perturb them into crossing orbits and violent impacts occur, thus gradually forming more massive objects. It is a common belief now that the Earth-type planets and rocky cores of the giant planets were formed by this two-stage process.

The question which is not very often addressed is where the boundary between these two stages occurs. This is an important issue, because the answer tells us the final mass of the objects which further evolve through chaotic collisional evolution, as well as the number of such bodies, and the conditions for which statistical treatments of collisional evolution are valid.

The standard paradigm for determining the embryo mass (Lissauer, 1987; Weidenschilling et al. 1997) presumes that planetary growth stops when the embryo "eats up" all the planetesimals within a "feeding zone" – an annulus of radial width of one Hill radius. Here the Hill radius is defined as

$$R_H = a\left(\frac{M_p}{M_c}\right)^{1/3}, \tag{1}$$

with a being the distance between the massive body with mass M_p and the central star with mass M_c. If the surface mass density of planetesimals is Σ_0, then this "isolation mass" is $M_p = M_{is} \sim 2\pi a R_H \Sigma_0$, which means that

$$M_{is} \sim \frac{(2\pi a^2 \Sigma_0)^{3/2}}{M_c^{1/2}} = 1\times 10^{26}\ \mathrm{g}\ \left(\frac{\Sigma_0 a^2}{2\times 10^{-6}\ M_\odot}\right)^{3/2}\left(\frac{M_c}{M_\odot}\right)^{-1/2}. \tag{2}$$

The estimate of $\Sigma_0 a^2$ is made for 1 AU and is based on standard assumptions for the protosolar nebula: $\Sigma_0 \approx 20(a/1\ \mathrm{AU})^{-3/2}\ \mathrm{g\ cm}^{-2}$, implying that ≈ 1 per cent of the minimum mass Solar nebula is contained in solids (Hayashi 1981).

The usual assumption is that before the mass of the largest protoplanet reaches M_{is}, the distribution of planetesimals is basically homogeneous. In some cases this is not a reasonable assumption. In particular Ida & Makino (1993) demonstrated using N-body simulations that a protoplanet could scatter planetesimals strongly if it is massive enough and, thus, clear a zone around it which is free from any solid bodies. The mass required to clear a gap in this way is not simply related to M_{is}. Kokubo & Ida (1998) later emphasized the importance of rapid heating of the planetesimal population by the forming planet in slowing down the subsequent accretion of planetesimals.

The process of clearing a gap in the planetesimal disk around a massive body is analogous to gap formation in gaseous disks (Takeuchi et al. 1996), which results from a competition between viscous spreading of the disk and gravitational interactions with the protoplanet. In a planetesimal disk the role of viscosity is played by mutual scattering of planetesimals. One can easily estimate the planetary embryo mass determined by this process. Let us assume that the planetesimal random

velocities are small – the disk is cold. Let $\Omega = (GM_c/a^3)^{1/2}$ be the orbital angular velocity. If m_0 is the mass of each of two planetesimal and $r_H = a(2m_0/M_c)^{1/3}$ is the corresponding Hill radius, then the typical displacement in a close encounter of these planetesimals on circular orbits separated by $< r_H$ is $\sim r_H$, and the typical random velocity kick is Ωr_H. Similarly, planetesimals within R_H from the massive object get kicked by $\sim R_H$ with frequency $\sim \Omega(R_H/a)$ (the inverse of the synodic period). If they are not able to diffuse back this distance during the time interval between kicks by the embryo, then a gap forms.

The "viscous" spreading distance is $\sim r_H K^{1/2}$ during one synodic period, where K is the number of collisions of a given small body with other planetesimals between consecutive approaches to the massive body. Assuming that the thickness of the planetesimal disk is $\sim r_H$, one can easily see that $K \sim r_H^2(\Sigma_0/m_0)(a/R_H) = a^2\Sigma_0/(m_0 M_p M_c)^{1/3}$. This implies that a gap in the planetesimal disk opens when $R_H^2 \gtrsim r_H^2 K$, or when

$$\frac{M_p M_c^{1/3}}{f(v/\Omega r_H)\Sigma_0 a^2 m_0^{1/3}} \gtrsim 1, \tag{3}$$

or, alternatively, when

$$M_p > M_{crit} \sim f(v/\Omega r_H)\Sigma_0 a^2 \left(\frac{m_0}{M_c}\right)^{1/3}. \tag{4}$$

Here $f(v/\Omega r_H)$ is a dimensionless function characterizing the effect of the planetesimal velocity dispersion v; we expect $f(x) \sim 1$ for $x \lesssim 1$ and $f(x) \sim x^{-2}\ln x$ for $x \gg 1$ (see §4.2). If the mass given by equation (4) is smaller than the isolation mass given by equation (2), the accretion stops because the protoplanet forms a gap, rather than because it consumes all the bodies in its feeding zone. Assuming typical values for protoplanetary disks one can get

$$M_{crit} \approx 4 \times 10^{23}\ \mathrm{g}\ f(v/\Omega r_H)\ \left(\frac{\Sigma_0 a^2}{2 \times 10^{-6}\ M_\odot}\right)\left(\frac{m_0}{10^{21}\mathrm{g}}\right)^{1/3}\left(\frac{M_c}{M_\odot}\right)^{-1/3}. \tag{5}$$

It is obvious from this estimate that gap formation could be very important in slowing down planetary accretion.

In this paper we analytically study the process of clearing a gap around a massive body in a planetesimal disk. We use an approach to treating the surface density evolution that was first developed by Petit & Hénon in their seminal series of papers (1987a, 1987b, hereafter PH, 1988). In §2 and Appendix A we derive a generalized form of their evolution equation, including the fluxes produced by the protoplanet and those generated by mutual gravitational perturbations between the planetesimals of the swarm.

In §3 we describe the solutions of the evolution equations for cold planetesimal disks, and compare our results with those obtained using N-body simulations. We comment on the applicability of our findings to the planet formation in the early Solar System, and describe briefly the relation between the surface density and planetesimal velocity dispersion evolution in §4.

2. Derivation of the general equation.

All the following calculations assume a Keplerian disk, although they could be easily extended to the case of an arbitrary rotation law.

We consider a disk of bodies (we will refer to them as planetesimals, but these could be other bodies, such as planetary ring particles) with $N(m, r, t)dm = \Sigma(m, r, t)dm/m$ being the surface number density of particles with mass between m and $m + dm$, whose guiding centers move at a distance r from the central body. It is important to keep in mind that r is the guiding center radius rather than the instantaneous radius. The instantaneous surface number density can only be obtained if $\Sigma(m, r, t)$ is supplemented by the random velocity distribution of planetesimals.

We also assume that a single massive body with mass M_p moves on a circular orbit in this planetesimal swarm (we take orbit to have a fixed radius and we comment later on the effects of migration) and we assume that its mass is much larger than the masses of the individual swarm particles. The mass of the central body is M_c and the distance of the planet from the central body is a. We will also use the relative masses of the bodies with respect to M_c: $\mu_p = M_p/M_c$ for the planet and $\mu = m/M_c$ for the planetesimals with mass m.

The interactions between particles in the gravitational field of a central body are described by Hill's equations, which are valid in the limit $\mu, \mu_p \ll 1$, which is always true in problems which we will study. It was demonstrated by Hénon & Petit (1986) that in this case the motion of nearby gravitationally interacting particles can be separated into center-of-mass motion, which is invariant during the interaction, and relative motion. If one defines new dimensionless coordinates where all the distances and relative velocities are normalized by $a(\mu_1 + \mu_2)^{1/3}$, then the equations of relative motion of particles 1 and 2 do not depend on their masses in these coordinates. Let us set h to be the distance between the guiding centers of interacting particles in these coordinates and $P(h, \Delta h)d\Delta h$ to be the probability of having a change in h in the range $(\Delta h, \Delta h + d\Delta h$ in an encounter. Then $P(h, \Delta h)$ does not depend upon the masses of particles involved in a collision, but does depend on the random velocity distribution function of the planetesimals.

In Appendix A we derive the general equation of the surface density evolution, which in many aspects parallels the derivation of equation (44) in PH. Let us set $N_i = N(m_i, r, t)$. Let $A = (r/2)(d\Omega/dr)$ be the function determining the local shear, $A = -(3/4)\Omega$ for a Keplerian rotation law. Then the surface density evolution is given by

$$\frac{\partial N_1(r)}{\partial t} = -2|A|a^2 \int_0^\infty dm_2(\mu_1 + \mu_2)^{2/3} \int_{-\infty}^{\infty} dh|h| \Bigg\{ N_1(r)N_2[r - (\mu_1 + \mu_2)^{1/3} rh] - \int_{-\infty}^{\infty} d(\Delta h)P(h, \Delta h)N_1[r + D(\Delta h)]N_2[r + D(\Delta h) - (\mu_1 + \mu_2)^{1/3} ah] \Bigg\}, \quad (6)$$

where

$$D(\Delta h) = -\frac{\mu_2 a}{(\mu_1+\mu_2)^{2/3}}\Delta h. \tag{7}$$

Note that the factor r is replaced with a in equations (6),(7), where it is appropriate, because at this level of approximation there is no difference between them, since they are both much larger than the Hill radius.

This differs from the equation derived in PH because it does not assume that surface density varies slowly on scales of the order of the Hill radius. If we made this assumption and expanded $N(m,r,t)$ up to the second order in h locally we would reduce equation (6) to the one derived in PH, equation (44), which we reproduce here:

$$\frac{\partial N_1}{\partial t} = |A|r^4 \int\limits_0^\infty dm_2 \left[2I_1\mu_2(\mu_1+\mu_2)^{1/3}\frac{\partial}{\partial r}\left(N_1\frac{\partial N_2}{\partial r}\right) + I_2\frac{\mu_2^2}{(\mu_1+\mu_2)^{2/3}}\frac{\partial^2(N_1N_2)}{\partial r^2}\right]. \tag{8}$$

Here I_1 and I_2 are dimensionless moments of the probability distribution $P(h,\Delta h)$:

$$I_1 \equiv \int\limits_{-\infty}^{\infty}|h|hdh\int\limits_{-\infty}^{\infty}d(\Delta h)\Delta h P(h,\Delta h) = \int\limits_{-\infty}^{\infty}|h|hdh\langle\Delta h\rangle, \tag{9}$$

$$I_2 \equiv \int\limits_{-\infty}^{\infty}|h|dh\int\limits_{-\infty}^{\infty}d(\Delta h)(\Delta h)^2 P(h,\Delta h) = \int\limits_{-\infty}^{\infty}|h|dh\langle(\Delta h)^2\rangle, \tag{10}$$

and symmetry dictates that

$$\int\limits_{-\infty}^{\infty}|h|dh\int\limits_{-\infty}^{\infty}d(\Delta h)\Delta h P(h,\Delta h) = 0. \tag{11}$$

For a cold disk it was demonstrated by PH that

$$I_1 = -3.07, \quad I_2 = 17.72. \tag{12}$$

We can now easily include the effect of a massive body on the surface density evolution. To do this we take surface density to consist of two parts: one representing a continuous distribution of small masses, corresponding to planetesimals, and another arising from the massive body. One can write the contribution from the embryo in the following form:

$$N_{em}(m,r,t) = \frac{1}{2\pi a}\delta(m-M_p)\delta(r-a). \tag{13}$$

We neglect migration of the embryo, so we assume a is fixed.

Substituting (13) into (6) and assuming that the embryo is much more massive than any of the planetesimals, $M_p \gg m$, we get

$$\frac{\partial N_1}{\partial t} = -2Aa^2 \int\limits_0^\infty dm_2(\mu_1+\mu_2)^{2/3} \int\limits_{-\infty}^{\infty} dh|h| \Bigg\{ N_1(r) N_2[r-(\mu_1+\mu_2)^{1/3}ah]$$
$$- \int\limits_{-\infty}^{\infty} d(\Delta h) P(h,\Delta h) N_1[r+D(\Delta h)] N_2[r+D(\Delta h)-(\mu_1+\mu_2)^{1/3}ah] \Bigg\} \tag{14}$$
$$-\frac{A}{\pi a} \left\{ N_1(r)|r-a| - \frac{1}{\mu_p^{1/3}a} \int\limits_{-\infty}^{\infty} dr_1 N_1(r_1)|r_1-a| P\left(\frac{r_1-a}{a\mu_p^{1/3}}, \frac{r-r_1}{a\mu_p^{1/3}}\right) \right\}.$$

We can make other simplifications taking the following into account. In our particular problem the relevant length scale for any structure is the Hill radius of the massive body R_H. Planetesimals can get kicks when interacting with the massive body which change their guiding centers by $\sim R_H$. At the same time mutual interactions between the planetesimals are unable to produce such large displacements (since $\mu_p \gg \mu_i$). One can thus assume that for planetesimal-planetesimal interactions the surface density varies only slowly on the scale $r_H \ll R_H$, and locally expand the first part of the r.h.s. of equation (6) in a Taylor series in h, as was done in equation (8). At the same time the second part of the r.h.s., representing the interaction with the large body, cannot be simplified in a similar way.

We also make some additional changes: we move the origin of r to a (simply set $r - a = r'$) and switch from r' to a dimensionless distance from the planetary embryo $H = r'/(\mu_p^{1/3}a)$. Then we get

$$\frac{\mu_p^{2/3}}{Aa^2}\frac{\partial N_1(H)}{\partial t} = \int\limits_0^\infty dm_2 \left[2I_1\mu_2(\mu_1+\mu_2)^{1/3} \frac{\partial}{\partial H}\left(N_1\frac{\partial N_2}{\partial H}\right) + I_2 \frac{\mu_2^2}{(\mu_1+\mu_2)^{2/3}} \frac{\partial^2(N_1N_2)}{\partial H^2} \right]$$
$$-\frac{\mu_p}{\pi a^2}\left[N_1(H)|H| - \int\limits_{-\infty}^{\infty} dH_1 \; N_1(H_1)|H_1| P(H_1, H-H_1) \right]. \tag{15}$$

In deriving this form of the evolution equation we only assumed that $\mu_p \gg \mu_1$. So, this nonlinear integro-differential equation can adequately describe the evolution of the surface density of planetesimals in the disk-protoplanet system.

2.1. Single mass planetesimals.

The constituent bodies of planetesimal disks are likely to have quite a broad range of masses. However, right now we are going to concentrate on the simple case of single mass planetesimals, that

is we assume all planetesimals to have a unique mass m_0. Then $N(m, H) = \sigma(H)(\Sigma_0/m_0)\delta(m - m_0)$, where Σ_0 is the surface mass density of particles at infinity which we take to be a reference value (it follows then that $\sigma(\infty) = 1$). Substituting this assumption into equation (15) and performing an integral over m_2 one obtains that

$$\frac{1}{I}\frac{\partial\sigma}{\partial\tau} = \frac{\partial^2\sigma^2}{\partial H^2} - \lambda\left[\sigma(H)|H| - \int\limits_{-\infty}^{\infty} dH_1\ \sigma(H_1)|H_1|P(H_1, H - H_1)\right], \tag{16}$$

where $\mu_0 = m_0/M_c$ and

$$I \equiv I_1 + \frac{I_2}{2} = \frac{1}{2}\int\limits_{-\infty}^{\infty} |h|\langle 2h\Delta h + (\Delta h)^2\rangle dh. \tag{17}$$

The new time variable τ is defined as

$$\tau = \frac{t}{t_0}, \qquad \text{where} \qquad t_0 = \frac{\mu_p^{2/3} M_c}{2^{1/3} A \mu_0^{1/3} \Sigma_0 a^2}, \tag{18}$$

and [cf. eq. (3)]

$$\lambda = \frac{M_p}{2^{1/3}\pi\Sigma_0 a^2 I \mu_0^{1/3}}. \tag{19}$$

One should notice that the first term in the r.h.s. of equation (16) and the expression in brackets are both dimensionless; all of the dimensional information is hidden in λ and τ.

For a cold disk (rms velocity dispersion of planetesimals in r-direction $v_r \ll \Omega r_H$) we obtain using (12) that

$$I = 5.79, \tag{20}$$

and, thus,

$$\lambda = 0.0436\ \frac{M_p}{\Sigma_0 a^2 \mu_0^{1/3}}, \qquad v_r \ll \Omega r_H. \tag{21}$$

For a hot disk ($v_r \gg \Omega r_H$) one has [see the discussion after equation (33) in §4.2]

$$I = 29.8\ \frac{\Omega^2 r_H^2}{v_r^2} \ln\Lambda, \quad \text{with} \quad \Lambda \sim \left(\frac{v_r^2}{\Omega^2 r_H^2}\right)^{3/2}. \tag{22}$$

It is assumed here that the ratio of vertical to radial random velocity dispersions in a planetesimal disk is equal to 0.5. Using (22) one finds that

$$\lambda = 0.0085\ \frac{M_p}{\Sigma_0 a^2 \mu_0^{1/3}} \frac{v_r^2}{\Omega^2 r_H^2} (\ln\Lambda)^{-1}, \qquad v_r \gg \Omega r_H. \tag{23}$$

In the intermediate regime, when $v_r \sim \Omega r_H$, there is no analytic expression for I and one has to interpolate between the two asymptotic behaviors given by (20) and (22).

The parameter λ quantifies the influence of the planetary perturbations on the uniformity of the planetesimal disk. The first term on the r.h.s. of (16) describes the nonlinear diffusion of particles due to mutual gravitational scattering, and tends to iron out any initial inhomogeneities. The second term represents the effect of the planet, which tends to carve out a gap in the distribution of planetesimals. The steady state originates when these two effects balance each other.

When the protoplanetary mass is large, λ is also large and the expression in brackets dominates the evolution. It leads to gap formation. On the contrary, if the planetary mass is small we can neglect the corresponding term in the r.h.s. of (16) and obtain a nonlinear diffusion equation, which drives the planetesimal distribution towards a homogeneous state. So, one can say that a gap (or at least a significant depression in the surface density of the planetesimals) is formed when $\lambda \gtrsim 1$. From equations (16) - (19) one can derive the characteristic time required for a gap to form when $\lambda \gtrsim 1$:

$$t_{open} = \frac{t_0}{I\lambda} = \frac{\pi}{|A|\mu_p^{1/3}} = \frac{2T}{3\mu_p^{1/3}} \approx 400\ T\ \left(\frac{M_p}{10^{25}\ \mathrm{g}}\right)^{-1/3} \left(\frac{M_c}{M_\odot}\right)^{1/3}, \tag{24}$$

where $T = 2\pi/\Omega$ is the orbital period of the embryo. Note that t_{open} is approximately the synodic period of a body at R_H from the embryo and, thus, is independent of the planetesimal mass m_0, the surface density Σ, and the numerical factor I.

In §3 we confirm these arguments by solving equation (16) numerically.

3. Numerical results.

3.1. Solution of the equation of evolution.

We solved equation (16) in a simplified setting, in which we completely neglect the velocity evolution of the planetesimal population. Thus, we assume the integral I embodying all the kinetic properties of planetesimals to be fixed in time. For simplicity we set $I = 1$; this only affects the timescale of the gap formation by a constant factor and does not change the evolution at all.

We also assume that planetesimals have very small random motion on the scale of the embryo's Hill radius: $v \ll \Omega R_H$. This simplifies our treatment a lot, because in this case scattering is deterministic so that $P(h, \Delta h) = \delta(\Delta h - h'(h) + h)$. The behavior of the function $h'(h)$ which gives the final semimajor axis difference as a function of the initial difference was described in detail by PH. For the sake of convenience we reproduce this dependence in Appendix B. Also, in this approximation, the instantaneous surface number density is given simply by $\sigma(H)$, because the guiding center and instantaneous radii coincide. The assumption of a cold disk might be reasonable in cases such as the early stages of gap clearing in a planetesimal disk, when scattering by the embryo could be in the shear-dominated regime, or all the time in dense planetary rings (Petit & Hénon 1987a).

We solved equation (16) with periodic boundary conditions, simply assuming that $\partial\sigma/\partial H = 0$

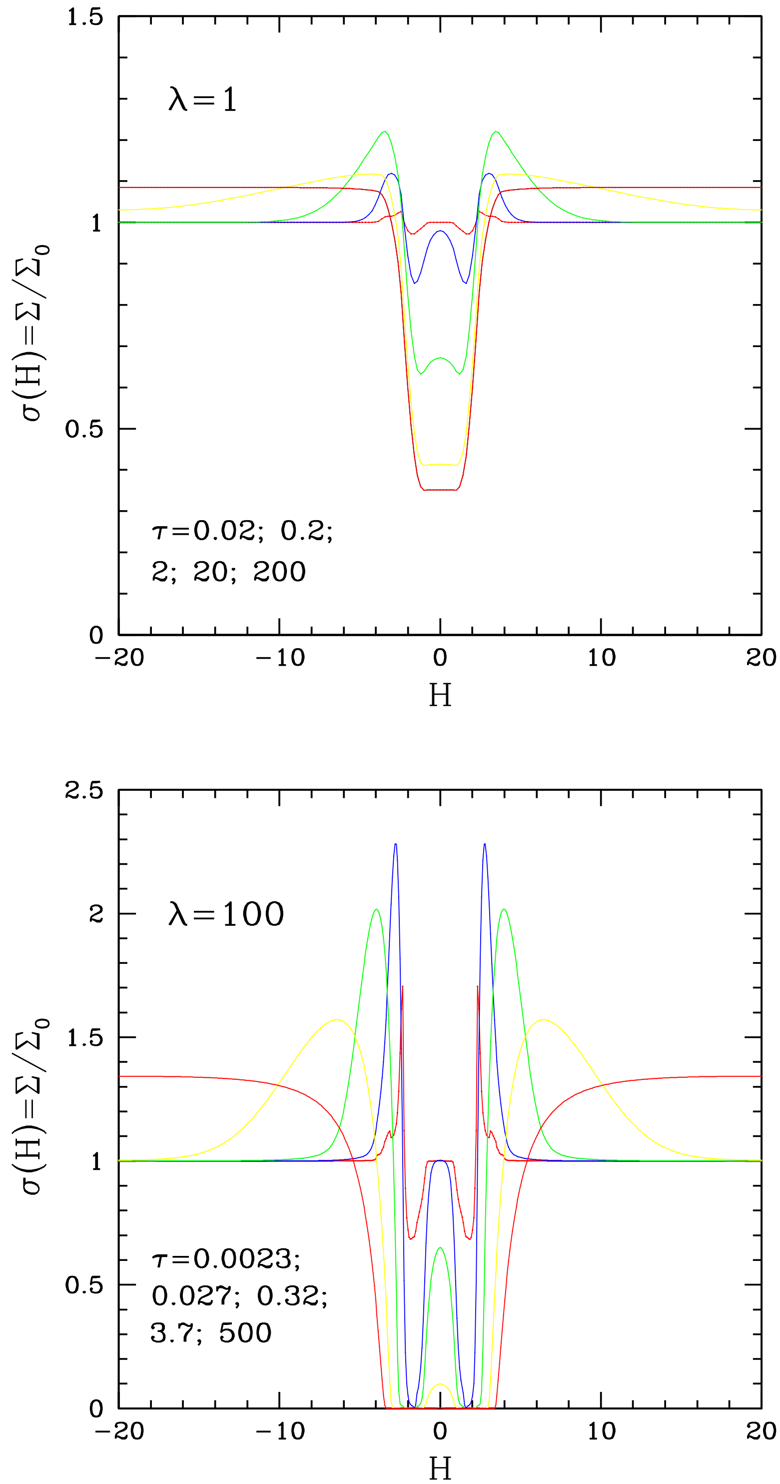

Fig. 1.— The time evolution of the surface density in a cold planetesimal disk with a single massive body at $H = 0$, for two values of the parameter λ defined in equation (19). In the *top panel* the case $\lambda = 1$ is described, in the *bottom panel* we consider $\lambda = 100$. The dimensionless time τ is indicated on the panels; larger τ corresponds to a deeper gap. Notice the presence of a bump at the center of a forming gap which is due to particles in horseshoe orbits near the massive body. The increase in asymptotic surface density at late τ is an artifact of the use of periodic boundary conditions at $H = \pm 20$, which forces $\int_{-20}^{20} \sigma(H) dH$ to be conserved.

at $H = \pm L$. The functional form of $P(h, \Delta h)$ is given by equation (B1). We usually assume $L = 20.0$ here (in units of the Hill radius of planet) and take the initial surface number density to be constant: $\sigma(H, 0) = 1$.

In Figure 1 we show the evolution of the surface density with time [we use the dimensionless time τ given by equation (18)] for $\lambda = 1$ and 100. In the first case a gap is never actually formed and in the steady state there is only a density depression around the planet. Thus the particles can be still accreted by the protoplanet, but the efficiency of this process is reduced.

In the case $\lambda = 100$ the gap is formed very quickly, which is in general agreement with the expected timescale for gap formation ($\tau_{open} \sim \lambda^{-1}$ in this case), although it takes some time after that for the density distribution to settle to a steady state.

In both cases one should notice a bump inside the gap which decays with time. It corresponds to the horseshoe orbits in the immediate vicinity of an embryo. This is in agreement with Monte-Carlo and N-body simulations performed earlier (Petit & Hénon, 1988; Tanaka & Ida, 1997; Spahn & Sremčević 2000).

In Figure 2 we show the final state of the surface density for several values of λ, so that one can see that gap gets deeper and wider as λ increases. It also looks like the condition $\lambda = 1$ is a reasonable approximate criterion for gap formation.

3.2. Comparison with N-body simulations.

We may compare our analytical results from equation (19) with N-body simulations by Ida & Makino (1993, hereafter IM93) and Tanaka & Ida (1997, hereafter TI97).

Figure 3 of IM93 and Figure 1 of TI97 show well developed gaps in a gas-free planetesimal disk. The gaps seen in N-body simulations are never clean because random motion of planetesimals is naturally included, and this permits some of them to be present in the gap. The parameters used in the production of these Figures correspond to $\lambda \approx 25/I$ in the first case while in the second $\lambda \approx 60/I$. In both cases the velocity dispersion of planetesimals is large ($v/\Omega r_H \sim 20 - 70$) and we expect $I \ll 1$ (see §4.2), meaning that $\lambda \gg 1$ and the condition for gap formation in the distribution of guiding centers of planetesimals should be fulfilled. It was also demonstrated by TI that gas drag could clear the gap of residual high-velocity planetesimals and thus stop accretion completely.

In Figure 6 of IM93 there is shown a sequence of scenarios for different ratios of the M_p/m_0 — planet to planetesimal masses. In the case $M_p/m_0 = 10$, when the gap is barely seen at all, $\lambda \approx 2.2/I$; since planetesimals are not strongly heated, presumably $\lambda \lesssim 1$. For $M_p/m_0 = 30$, when the gap becomes pronounced, $\lambda \approx 6.6/I$ and $\lambda \sim 1$. Finally for $M_p/m_0 = 100$, when the gap is quite significant, $\lambda \approx 22/I$; heating also becomes important and $\lambda \gg 1$. In addition, these results confirm our prediction below in §4.2 that for $\lambda \sim 1$, when gap starts to form, planetary perturbations begin to dominate planetesimal random velocity stirring within $\sim R_H$ of the planet.

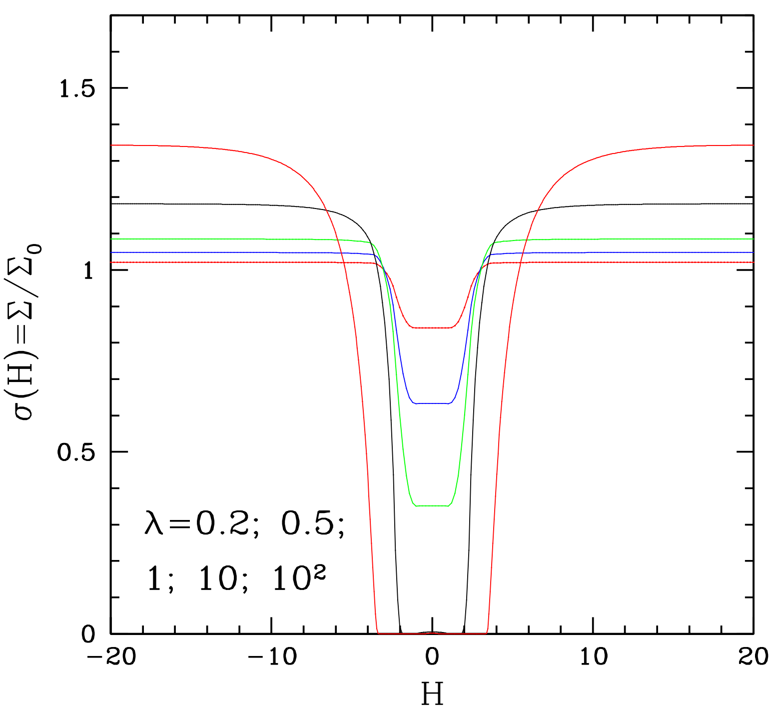

Fig. 2.— The final distribution of the surface density for a cold disk with a single massive body, for several values of the parameter λ: $0.2; 0.5; 1; 10; 100$. The higher values of λ correspond to progressively deeper gaps in the Figure. The increase in asymptotic surface density at late τ is an artifact of the use of periodic boundary conditions.

This comparison shows that estimates of gap formation based on the parameter λ are qualitatively consistent with N-body simulations.

4. Discussion

4.1. Applications.

We have established the usefulness of the parameter λ in describing gap formation. Now we are going to use it to determine the mass of the body which could open a gap in a planetesimal disk. This is simply done by setting $\lambda = 1$. We rewrite this condition as

$$M_{crit} = 2^{1/3}\pi I \Sigma_0 a^2 \left(\frac{m_0}{M_c}\right)^{1/3}, \tag{25}$$

where M_{crit} is the planet mass for which $\lambda = 1$. One can easily see that this value of critical planetary mass coincides with our simple estimate in equation (4) if we set $f(v/\Omega r_H) = 2^{1/3}\pi I$.

As we mentioned in §1, it is usually assumed that accretion stops when planet hoovers up a zone in the planetesimal disk with width equal to its Hill radius [eq. (2)]. Then the ratio of the two critical masses is

$$\frac{M_{crit}}{M_{is}} = \frac{I}{2^{7/6}\pi^{1/2}}\left(\frac{m_0}{\Sigma_0 a^2}\right)^{1/3}\left(\frac{M_c}{\Sigma_0 a^2}\right)^{1/6}. \tag{26}$$

We expect the total mass of the circumstellar disk to be of order $0.1 - 0.01$ of the mass of the central star M_c (Osterloh & Beckwith 1995; Mannings & Sargent 2000). For the protosolar nebula it is often assumed that surface density of gas at 1 AU is ~ 2000 g cm^{-2} (Hayashi 1981). The mass fraction of heavy elements, which contribute to solid body formation in this disk, is ~ 0.01. Using this information we can estimate that $M_c/(\Sigma_0 a^2) \sim 10^5 - 10^6$ at 1 AU and the corresponding factor does not contribute a lot to the ratio in (26) (it varies roughly from 7 to 10). The factor $2^{-7/6}\pi^{-1/2} I \approx 1.5$ for a cold disk [see eq. (20)], and is significantly smaller for hot disks [it is likely to be $\sim (m_0/M_p)^{2/3}$, see §4.2].

If we take planetesimals to be rocky bodies with radius ~ 50 km and mass $\sim 10^{21}$ g, then $m_0/(\Sigma_0 a^2) \sim 10^{-6} - 10^{-7}$. In the end we obtain that

$$M_{crit}/M_{is} \sim (10^{-1} - 10^{-2}). \tag{27}$$

This result means that accretion is slowed down long before the clearing of the feeding zone. We conclude that the masses of planetary embryos are ~ 10 to 100 times smaller than predicted by arguments based on clearing the feeding zone [see eq. (2)].

4.2. Random motions of planetesimals

Equation (16) fully describes the surface density evolution (or its steady-state structure) only if it is supplied with information about random motions of particles within the planetesimal disk, which determine $P(h,\Delta h)$ and its moment I.

For a cold disk (zero velocity dispersion) I is given by (20). In our case the planetesimal swarm is unlikely to be cold, because planetesimals will be scattered by the planet, and also will scatter each other. An important point to note here is that these two types of scattering probably operate in quite different regimes. Indeed, the natural parameter determining the heating regime is the ratio of the velocity dispersion to the shear across the Hill radius. For the scattering by the planet this parameter is $S \sim v/\Omega R_H$, with $R_H = a(M_p/M_c)^{1/3}$ while for the interaction with other planetesimals it is $s \sim v/\Omega r_H$, with $r_H = a(2m_0/M_c)^{1/3}$, so that $s \sim S(M_p/m_0)^{1/3}$. Thus, scattering by the planetary embryo could be in a shear-dominated regime, while mutual scattering of planetesimals is practically always in a dispersion-dominated one.

During a passage within a Hill radius from a massive body, a planetesimal initially on a circular orbit gets a significant kick, so that its velocity dispersion increases by $\sim \Omega R_H$. Thus, the part of the disk within a Hill radius from the planetary embryo will be heated to $v \sim \Omega R_H$ corresponding to $S \sim 1$ in time $\sim \Omega^{-1}(a/R_H) = \Omega^{-1}\mu_p^{-1/3}$. Thus the stirring rate by planetary scattering is

$$\left.\frac{ds^2}{dt}\right|_{pl} \sim \Omega\left(\frac{M_p^3}{m_0^2 M_c}\right)^{1/3}, \qquad \text{for} \qquad S \lesssim 1, \tag{28}$$

within a radial distance R_H from the embryo. The same kind of estimate could be derived for the excitation of the vertical random motions.

This heating rate quickly leads to $s \gg 1$ and, thus, the self-heating of the planetesimal population should be calculated in a dispersion-dominated regime. In particular the integral I in equation (16) describing the scattering of planetesimals by their mutual interactions should be calculated in this approximation.

Stewart & Ida (2000) considered velocity stirring in the dispersion-dominated regime. Their results demonstrate that the horizontal stirring rate is

$$\left.\frac{ds^2}{dt}\right|_{self} = \Omega\frac{\Sigma_0}{m_0} r_H^2 \langle P_{VS}\rangle, \tag{29}$$

if the vertical velocity dispersion is of the same order as a horizontal one (as we normally expect), and they provide a closed analytic form for the stirring coefficient $\langle P_{VS}\rangle$ as a function of planetesimal velocity dispersion. The coefficient $\langle P_{VS}\rangle$ is defined to represent the change in the square of the relative eccentricity [see Hénon & Petit (1986)], averaged over the vertical and horizontal orbital phases, and the velocity distribution of planetesimals (Ida, 1990; Stewart & Ida, 2000):

$$\langle P_{VS}\rangle = \int \Delta e_u^2(e_u, i_u, h, \tau, \omega) f(e_u, i_u) de_u^2 di_u^2 \frac{3}{2}|h| dh \frac{d\tau d\omega}{4\pi^2}, \tag{30}$$

where e_u, i_u, τ, ω are the relative eccentricity, inclination, horizontal and vertical orbital phases, h is the separation of semimajor axes of the interacting particles and $f(e_u, i_u)$ is the corresponding distribution function. We assume that e_u, i_u, and h are properly normalized by the Hill radius for particles participating in the collision [in this case our definition of $\langle P_{VS}\rangle$ differs from Stewart & Ida (2000) by a factor of $(m_1 + m_2)^{4/3}/(3M_c)^{4/3}$, where m_1 and m_2 are the masses of interacting planetesimals].

Vertical stirring is described by a formula similar to (29) but with a different stirring coefficient

$$\langle Q_{VS}\rangle = \int \Delta i_u^2(e_u, i_u, h, \tau, \omega) f(e_u, i_u) de_u^2 di_u^2 \frac{3}{2}|h|dh\frac{d\tau d\omega}{4\pi^2}. \tag{31}$$

Now we can say more about the relation of the integral I to the kinetic properties of the planetesimal population. One can easily see that the averaging over all possible Δh for a given initial h used in definition (17) is equivalent to averaging over the vertical and horizontal orbital phases at a given relative eccentricity and inclination and then over the distribution of relative eccentricities and inclinations, that is

$$I = \frac{1}{2}\int\limits_{-\infty}^{\infty} |h|dh \int f(e_u, i_u) de_u^2 di_u^2 \frac{d\omega d\tau}{(2\pi)^2}[2h\Delta h + (\Delta h)^2]. \tag{32}$$

From the conservation of Jacobi constant one has $2h\Delta h + (\Delta h)^2 = (4/3)[\Delta(e_u^2) + \Delta(i_u^2)]$. Then, substituting into (32) we get that

$$I = \frac{4}{9}\left(\langle P_{VS}\rangle + \langle Q_{VS}\rangle\right). \tag{33}$$

This is an important result, because it relates the evolution of the surface density of a spatially inhomogeneous planetesimal population to the viscous stirring in a homogeneous disk.

Using these expressions we can determine the relative role of self-heating of the disk and planetary heating in the vicinity of the massive body. Of course, the former dominates beyond several R_H from planet, but within $\sim R_H$ of the embryo one can get from equations (19), (28), (29), and (33) the simple result that

$$\frac{(ds^2/dt)|_{pl}}{(ds^2/dt)|_{self}} \sim \lambda. \tag{34}$$

This means that for $\lambda \gtrsim 1$, when a gap starts to form, embryo perturbations begin to dominate planetesimal heating within $\sim R_H$ of the planet, i.e. when the planet dominates the surface density evolution it also dominates the heating.

The evolution of kinetic properties of planetesimals could be neglected if there is an effective velocity damping due to inelastic collisions between bodies, as in the case of planetary rings (Petit & Hénon, 1987a), or if there is a strong gas drag. In the planetesimal case, however, gravitational

stirring dominates over damping (Kenyon & Luu 1998) and then the fact that a gap in the distribution of guiding centers is opened does not automatically mean that planetesimals cannot reach the planet, because in the course of scattering their velocity dispersion grows as well [see equation (34)]. However, the accretion rate will drop anyway at least because of the less pronounced focussing (Safronov, 1972; Dones & Tremaine, 1996). Also, inhomogeneous random velocity evolution or gas drag could remove the residual planetesimals from the forming gap, thus bringing their surface density around the embryo to zero and shutting down accretion completely (see TI for N-body simulations in the presence of the gas drag). For this reason we believe that our results with no velocity evolution are applicable to the problem of planet accumulation in many cases.

All the stirring coefficients are functions of the vertical and horizontal velocity dispersions of planetesimal population. Stewart & Ida (2000) have in particular shown that $\langle P_{VS}\rangle, \langle Q_{VS}\rangle \propto s^{-2}\ln s$ for $s \gg 1$ [we used this result to derive equation (22)]. It means that to close properly the problem we need to couple equations for the velocity evolution to equation (16). In doing so, one should bear in mind that random motions are highly nonuniform in space, since stirring by the massive body is strongly localized within several Hill radii from it. Also, it is not clear that the velocity distribution of planetesimals can be adequately described by the Schwarzschild distribution, as is usually assumed for simplicity. For this reason we will not pursue this subject here and postpone its more detailed exploration to a future work.

It is important to note however, that $I \propto s^{-2}\ln s$ and $\lambda \propto s^2/\ln s$ for $s \gg 1$, as follows from the equations (22) and (23). Thus, as λ grows and the embryo heats up planetesimal population around it, the planetesimal "viscosity" decreases (increasing λ even more through this velocity coupling), which facilitates gap opening. This only strengthens our conclusion that a gap must form when the condition $\lambda \gtrsim 1$ is fulfilled, even without knowing the details of the random velocity evolution of planetesimals.

4.3. Effects of planetary migration.

In §3 we studied gap opening around a massive body, assuming that the background distribution of surface density of planetesimals is symmetric with respect to the position of the embryo. In this case we assumed that the embryo is fixed in radius and there is no migration at all.

It is more than likely that in real protoplanetary disks there are significant surface density gradients, which could drive embryo migration. This could in principle introduce significant changes into our picture. Indeed, if the embryo is able to migrate quickly it may move out of the gap it starts to form and, thus, gap formation would be suppressed.

Planetary migration will naturally occur in the course of embryo accumulation, since in the process of scattering planetesimals, the massive body exchanges its angular momentum with them, which leads to its migration. Indeed, planetesimals passing within R_H from the embryo get displaced by a distance $\sim R_H$, thus an embryo itself is displaced by $\sim R_H(m_0/M_p)$. During the time

interval Δt approximately $(\Sigma_0/m_0)R_H^2\Omega\Delta t$ planetesimals pass within embryo's Hill sphere. The surface number densities on both sides will likely be different by $\sim (\Sigma_0/m_0)(R_H/a)$, although it is likely that as the embryo moves in some direction it plows planetesimals in front of it, leaving behind a depression, which would tend to oppose the migration [see Ward & Hourigan (1989) for a similar effect in gaseous disks]. Thus, our previous assumption about the surface density difference is likely to be an upper limit and the actual migration will be weaker. One can easily calculate that the rate of this "maximum" migration is

$$\frac{1}{R_H}\frac{da}{dt} = \frac{dh}{dt} \sim \Omega\frac{\Sigma_0 a^2}{M_c}. \tag{35}$$

The time it takes the embryo to migrate through a zone with a width equal to its own Hill radius is then $\Omega^{-1}M_c/(\Sigma_0 a^2)$ and should be longer than t_{open} given by equation (24) if a gap is to be maintained. Thus, the necessary condition here is

$$\left(\frac{M_p}{M_c}\right)^{1/3} > \frac{\Sigma_0 a^2}{M_c}. \tag{36}$$

If, say, $\Sigma_0 a^2/M_c = 10^{-5}$, (and $\lambda \gtrsim 1$) then all bodies with masses $\gtrsim 10^{18}$ g will open a gap faster than they migrate through it.

Another type of migration could arise if the whole system is immersed in a massive gaseous disk, as should be the case in the early stages of protoplanetary evolution. In this case Goldreich & Tremaine (1980) demonstrated that it takes time $\sim \Omega^{-1}(h^2\Delta a/a^3)(M_c^2/M_p\Sigma_g a^2)$ for a planet to migrate a distance Δa in the radial direction, where h is the disk thickness, determined by its temperature, and Σ_g is the surface density of gaseous disk. In our case, the relevant lengthscale is again $\Delta a \sim R_H$, thus migration timescale is

$$t_{mig} \sim \Omega^{-1}\mu_p^{-2/3}\frac{M_c}{\Sigma_g a^2}\frac{h^2}{a^2}. \tag{37}$$

Comparing these two timescales we obtain:

$$\frac{t_{open}}{t_{mig}} \sim \mu_p^{1/3}\frac{\Sigma_g a^2}{M_c}\frac{a^2}{h^2} \sim 10^{-3}\left(\frac{M_p}{10^{24}\ \mathrm{g}}\right)^{1/3}\frac{\Sigma_g a^2}{10^{-3}\ M_\odot}\left(\frac{M_c}{M_\odot}\right)^{-4/3}\left(\frac{a/h}{30}\right)^2. \tag{38}$$

Thus, migration due to the interaction with a gaseous disk is unlikely to have an important effect on gap opening.

5. Summary

We studied the possibility that planetary formation due to the accretion of planetesimals could be significantly slowed or even stopped, due to gap formation around a forming planetary embryo, caused by the strong gravitational perturbations of planetesimals in its vicinity. We find a critical

parameter λ which describes the importance of gap formation [eq. (19)]. Predictions based on this parameter were compared with numerical N-body simulations of this process (IM93, TI97), and they are in good agreement.

Only the case of a single mass distribution of the particles in a disk was studied here. But it is plausible that our basic results hold true even if a distribution of masses exists. Only the characteristic planetesimal mass entering this parameter should be chosen carefully, and this question merits further investigation [see Kokubo & Ida (1996), (1998) for some numerical results].

Our findings were confirmed by solving the evolution equation (16) neglecting the velocity dispersion evolution of planetesimals in the disk. The kinematic properties of the planetesimal population are important in this sort of study, and the surface density evolution and velocity evolution of planetesimals are closely related. Nevertheless, we hope to have grasped the main qualitative features of the evolution of the distribution of guiding centers even without keeping track of the velocity evolution. The instantaneous density of planetesimals depends on their kinematic properties and should experience at least a decrease by a factor of several in the vicinity of the embryo, leading to slowing down the accretion (and gas drag and inhomogeneous velocity evolution could clear out the gap completely). We also stress that our results for a cold disk provide an upper limit to the embryo mass required to open a gap.

If the disk is not uniform, migration of the embryo itself also likely to occur in the course of its interaction with planetesimals or with the more massive gaseous disk from which the whole embryo-planetesimal system originally condensed. We have estimated how this could affect our results and show that gap formation is likely to occur even when migration is present.

Finally, the evolution equation itself, coupled with the equations of planetesimal velocity evolution, provide us with a powerful tool to study the formation and evolution of planetary embryos. Since our results seem to be in good agreement with N-body simulations, we may use this machinery for other problems of a similar nature. An obvious example is the coupled evolution of several planetary embryos.

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A. Derivation of equation (6).

Following Petit & Hénon (1987b), we first calculate the current of particles with mass m_1 (initially located at the guiding-center radius r_1) through the circle of radius r, due to the interaction with a single particle of mass m_2, located at guiding-center radius r_2. The characteristic dimensionless relative distance for the two particles involved in this interaction

$$h = \frac{|r_1 - r_2|}{r(\mu_1 + \mu_2)^{1/3}}. \tag{A1}$$

It was demonstrated by Hénon & Petit (1986) that in coordinates normalized in this way, the equations of relative motion of particles do not depend upon their masses. For the particle m_1 initially located to the left of the boundary at r, to cross it to the right one needs

$$\Delta h > \Delta h_{min} = \frac{|r - r_1|}{r} \frac{m_1 + m_2}{m_2(\mu_1 + \mu_2)^{1/3}}, \tag{A2}$$

where the factor $(m_1 + m_2)/m_2$ arises because h describes relative motion of particles.

Then the left-to-right part of this flow of particles in the mass interval $(m_1, m_1 + dm_1)$ during the time dt is obviously given by

$$\int\limits_{-\infty}^{r} dr_1 \; 2|A||r_1 - r_2| \int\limits_{\Delta h_{min}}^{\infty} d(\Delta h) \times 2\pi r_1 P(h, \Delta h) N(m_1, r_1, t) dt \; dm_1. \tag{A3}$$

Here $|A||r_1 - r_2|$ is the local shear velocity between the interacting particles.

Summing over all possible positions and masses of particles m_2 we obtain the total left-to-right flow of particles in the range $(m_1, m_1 + dm_1)$:

$$\begin{aligned} \langle \Delta J_+ \rangle = dm_1 \int\limits_0^{\infty} dm_2 \int\limits_{-\infty}^{\infty} dr_2 \int\limits_{-\infty}^{r} dr_1 \int\limits_{\Delta h_{min}}^{\infty} d(\Delta h) \\ \times N(m_1, r_1, t) N(m_2, r_2, t) P(h, \Delta h) 4\pi r_1 |A||r_1 - r_2| dt, \end{aligned} \tag{A4}$$

This formula coincides with equation (28) of PH. We will further denote $f_i = f(m_i, ...), i = 1, 2$ for *any* function f for brevity, and replace the factor r_1 under the integral with r (since r_1 weakly varies on the scale of the Hill radius).

Making the change of variables from r_1, r_2 to R, h given by [cf. equations (30) and (31) of PH]

$$r_1 = r + R, \tag{A5}$$

$$r_2 = r + R - (\mu_1 + \mu_2)^{1/3} r h, \tag{A6}$$

one can reduce equation (A4) to

$$\begin{aligned} \langle \Delta J_+ \rangle = 4\pi |A| r^3 dt \; dm_1 \int\limits_0^{\infty} dm_2 (\mu_1 + \mu_2)^{2/3} \int\limits_{-\infty}^{\infty} dh \int\limits_{-\infty}^{0} dR \int\limits_{\Delta h_{min}}^{\infty} d(\Delta h) \\ \times N_1(r + R) N_2[r + R - (\mu_1 + \mu_2)^{1/3} r h] P(h, \Delta h) |h|. \end{aligned} \tag{A7}$$

We can change the order of integration over $d(\Delta h)$ and dR to get for $\langle \Delta J_+ \rangle$

$$\langle \Delta J_+ \rangle = 4\pi |A| r^3 dt\ dm_1 \int\limits_0^\infty dm_2 (\mu_1 + \mu_2)^{2/3} \int\limits_{-\infty}^\infty dh|h| \int\limits_0^\infty d(\Delta h) P(h, \Delta h) \int\limits_D^0 dR$$
$$\times N_1(r+R) N_2[r + R - (\mu_1 + \mu_2)^{1/3} rh], \tag{A8}$$

with

$$D = -\frac{\mu_2 r}{(\mu_1 + \mu_2)^{2/3}} \Delta h. \tag{A9}$$

Considering now the right-to-left flow of particles through r one can get for this component of flux

$$\langle \Delta J_- \rangle = 4\pi |A| r^3 dt\ dm_1 \int\limits_0^\infty dm_2 (\mu_1 + \mu_2)^{2/3} \int\limits_{-\infty}^\infty dh|h| \int\limits_{-\infty}^0 d(\Delta h) P(h, \Delta h) \int\limits_0^D dR$$
$$\times N_1(r+R) N_2[r + R - (\mu_1 + \mu_2)^{1/3} rh]. \tag{A10}$$

Now, the total flux of particles through the boundary at r is given by $\langle \Delta J \rangle = \langle \Delta J_+ \rangle - \langle \Delta J_- \rangle$. Then the equation of evolution is obtained by setting

$$\frac{\partial}{\partial t} \left[2\pi r N_1(m, r, t) \right] dt dm_1 = -\frac{\partial \langle \Delta J \rangle}{\partial r}. \tag{A11}$$

Here we do not differentiate r^3 in the right-hand side because it varies only weakly on the scale of Hill radius. Then the right-hand side of (A11) contains $\partial(N_1 N_2)/\partial r$ which obviously equals $\partial(N_1 N_2)/\partial R$. Taking this into account, one can trivially perform an integration over R in the r.h.s. of (A11) to obtain finally equation (6). In deriving it we have also taken into account that

$$\int\limits_{-\infty}^\infty d(\Delta h) P(h, \Delta h) = 1. \tag{A12}$$

One should note that in deriving (A7) we assumed that $\langle N_1 N_2 \rangle_\phi = \langle N_1 \rangle_\phi \langle N_2 \rangle_\phi$, where $\langle g \rangle_\phi$ means averaging quantity g over the azimuthal angle. This might not be true in the planetary disks which are cold and have a large viscosity (Spahn & Sremčević 2000), or during the initial stages of the gap development in planetesimal disk. However, we believe that it is unlikely to affect our results, since for hot disks and late times this separability assumption should be adequate. Thus, it is possible that our numerical results presented in §3 are somewhat different quantitatively at very early times from what a more detailed theory would predict. But we believe that our principal conclusions remain unchanged.

B. Form of $h(h')$ used in PH.

Petit & Hénon (1987b) have solved numerically the Hill equations in the case when the initial random motion of interacting particles is small. In this case the outcome of the interaction between two particles is deterministic, and they obtained the following form for the function $h(h')$, where h is the initial difference of semimajor axes of particles and h' is the final value of the same quantity, normalized by $a[(m_1+m_2)/M_c]^{1/3}$, where m_1 and m_2 are the masses of interacting bodies:

$$h'(h) = \begin{cases} h + 3.34377(h^5 + 0.2h^4 - 3.14h^3)^{-1}, & \text{if} \quad h \geq 1.75622, \\ 350h^2 - 1204h + 1038.5, & \text{if} \quad 1.75622 > h \geq 1.6777, \\ 2895.903h^5 - 20454.39h^4 + 57671.78h^3 & \\ -81146.35h^2 + 56984.57h - 15977.97, & \text{if} \quad 1.6777 > h \geq 1.2219, \\ -1832.5h^2 + 4361.35h - 2596.5, & \text{if} \quad 1.2219 > h \geq 1.17, \\ -h - 4.107085921(h^{-1} + 4h^4)\exp(-5.58505361h^3), & \text{if} \quad 1.17 > h \geq 0. \end{cases} \tag{B1}$$